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Note

Total choosability of planar graphs with maximum degree 4

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ABSTRACT

Let G be a planar graph with maximum degree 4. It is known that G is 8-totally choosable. It has been recently proved that if G has girth $g \geqslant 6$, then G is 5-totally choosable. In this note we improve the first result by showing that G is 7-totally choosable and complete the latter one by showing that G is 6-totally choosable if G has girth at least 5.

1. Introduction

A *total k-coloring* of a graph $G = (V, E)$ is a coloring of $V \cup E$ using at most k colors such that no two adjacent or incident elements get the same color. If G admits a total k-coloring, we say that G is *k-totally colorable*. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest integer k such that G is k-totally colorable. It is clear that $\Delta + 1 \leqslant \chi''(G)$, where Δ is the maximum degree of G. Vizing [9] conjectured that $\chi''(G) \leqslant \Delta + 2$ for every graph G. This conjecture was verified [1,6–8,10,11] for planar graphs with maximum degree $\Delta \neq 6$.

A *total-list-assignment* of G is a mapping L that assigns to every element $x \in V \cup E$ a set of colors $L(x)$. The graph G is *L-totally colorable* if G admits a total coloring c with $c(x) \in L(x)$ for any $x \in V \cup E$. Such a coloring is called an L-total coloring. If G is L-totally colorable for any total-list-assignment L with $|L(x)| \geqslant k$ for any $x \in V \cup E$, then G is *k-totally choosable*. The *list total chromatic number* of G, or *total choosability* of G, denoted by $\chi_l''(G)$, is the smallest integer k such that G is k-totally choosable. Borodin et al. [2] and Juvan et al. [5] independently conjectured that $\chi_l''(G) = \chi''(G)$ for any graph G. Juvan et al. [5] conjectured that $\chi_l''(G) \leqslant \Delta(G) + \mu(G) + 1$ for any multigraph G, where $\mu(G)$ is the multiplicity of G. In [4], it is proved that planar graphs with maximum degree Δ and girth g are $(\Delta + 2)$-totally choosable for $(\Delta, g) = (9, 3), (5, 4)$. The total choosability of planar graphs has been extensively studied with a number of results increasing with the maximum degree of the graphs. For $\Delta = 4$, very few results exist. Borodin et al. proved [2] that every planar graph with maximum degree 4 is 8-totally choosable. Recently, Chang et al. proved [3] that planar graphs with maximum degree 4 and girth $g \geqslant 6$ are 5-totally choosable. In this note, we improve Borodin's result and complete the latter result as follows.

The summation of the degree of all vertices of a graph is equal to twice its number of edges. So the *average degree* $\mathrm{ad}(G)$ of G is $\frac{2|E|}{|V|}$. We define the *maximum average degree* $\mathrm{mad}(G)$ of G as the maximum of the average degree of its subgraphs.

Theorem 1. *If G is a graph with maximum degree $\Delta = 4$ and maximum average degree $\mathrm{mad}(G) < \frac{10}{3}$, then G is 6-totally choosable.*

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Theorem 2. *If G is a planar graph with maximum degree $\Delta = 4$, then G is 7-totally choosable.*

2. Preliminaries

In this paper, we use the following notation. A k-, k^+- or k^--*vertex* is a vertex of degree k, at least k or at most k, respectively. The *degree* of a face f, denoted by $d(f)$, is the length of a boundary walk around f. In particular, a cut-edge is counted twice. A k-, k^+- or k^--*face* is a face of degree k, at least k or at most k, respectively. Let $f_k(v)$, $f_{k^+}(v)$ or $f_{k^-}(v)$ denote the number of k-faces, k^+-faces or k^--faces incident with a vertex v.

A graph H is *minimally non-k-totally choosable* if it is not k-totally choosable but all its proper subgraphs are k-totally choosable. Chang et al. proved [3] the following Lemma 3. We additionally prove Lemma 4.

Lemma 3. *The following hold for any minimally non-k-totally choosable graph H with maximum degree $\Delta \leqslant k - 1$.*

(a) *H is connected.*
(b) *If $e = uv$ is an edge in H with $d(u) \leqslant \frac{k-1}{2}$, then $d(u) + d(v) \geqslant k + 1$. In particular, $\delta(H) \geqslant k + 1 - \Delta$ and so H has no 1-vertex.*
(c) *H has no even cycle $v_1 v_2 \cdots v_{2t} v_1$ with $d(v_i) \leqslant \min\left\{\frac{k-1}{2}, k + 1 - \Delta\right\}$ for each odd i.*

Lemma 4. *In a minimally non-6-totally choosable graph H with maximum degree $\Delta = 4, 5$, any 3-vertex is adjacent to at most one 3-vertex.*

Proof. Suppose to the contrary that there is a 3-vertex u adjacent to two 3-vertices v_1 and v_2. Let v_3 be the other vertex adjacent to u and u_i, w_i be the other vertices adjacent to v_i for $i = 1, 2$. Suppose L is a total-list-assignment of H such that $|L(x)| \geqslant 6$ for any $x \in V \cup E$. By the minimality of H, $H - \{uv_1, uv_2\}$ has an L-total coloring c. Uncolor u. Note that each of uv_1, uv_2 touches at most four colors, so that they can be totally colored properly. Vertex u now touches at most six colors. Suppose there is no color available for u. We may assume $L(u) = \{1, 2, 3, 4, 5, 6\}$, $c(v_i) = i$, $c(uv_i) = 3 + i$, for $i = 1, 2, 3$. If there is a color $\alpha \in L(v_1) \setminus \{1, 4, c(u_1), c(w_1), c(u_1v_1), c(w_1v_1)\}$, recolor v_1 with color α and color u with color 1. Now assume $L(v_1) = \{1, 4, c(u_1), c(w_1), c(u_1v_1), c(w_1v_1)\}$. Similarly, we may assume $L(uv_1) = \{1, 4, 5, 6, c(u_1v_1), c(w_1v_1)\}$, $L(v_2) = \{2, 5, c(u_2), c(w_2), c(u_2v_2), c(w_2v_2)\}$ and $L(uv_2) = \{2, 4, 5, 6, c(u_2v_2), c(w_2v_2)\}$. Recolor v_1 and uv_2 with color 3, uv_1 with color 5 and u with color 1. $\square$

3. Proof of Theorem 1

Suppose the theorem is false. Let $H = (V, E)$ be a minimum counterexample. It was proved [5] that graphs with maximum degree less than 4 are 5-totally choosable. So H is minimally non-6-totally choosable. Lemma 3 implies that $\delta(H) \geqslant 3$.

We use a discharging procedure. The initial charge of a vertex v of H is $\omega(v) = d(v) - \frac{10}{3}$. The total charge of the vertices of H is equal to

$$\sum_{v \in V(G)} \left(d(v) - \frac{10}{3}\right) = |V(G)| \times \left(\mathrm{ad}(G) - \frac{10}{3}\right) < 0.$$

Every 3-vertex receives charge $\frac{1}{6}$ from each of its neighbors of degree 4. Let $\omega^*(v)$ be the final charge of a vertex v. If v is a 3-vertex, then $\omega(v) = -\frac{1}{3}$. If w is a 4-vertex adjacent to v, then v receives charge $\frac{1}{6}$ from v, and v is adjacent to at least two 4-vertices by Lemma 4. Thus $\omega^*(v) \geqslant \omega(v) + 2 \times \frac{1}{6} = 0$. Suppose v is a 4-vertex. Initially, $\omega(v) = \frac{2}{3}$ and v sends charge $\frac{1}{6}$ to each adjacent 3-vertex. Thus $\omega^*(v) \geqslant \omega(v) - 4 \times \frac{1}{6} = 0$. Hence the total charge of the vertices is nonnegative, a contradiction.

The *girth* of a planar graph is the length of a smallest cycle in the graph. For a planar graph with girth g, $\mathrm{mad}(G) < \frac{2g}{g-2}$, hence the following corollary.

Corollary 5. *If G is a planar graph with maximum degree $\Delta = 4$ and girth $g \geqslant 5$, then G is 6-totally choosable.*

4. Proof of Theorem 2

Suppose the theorem is false. Let $H = (V, E)$ be a minimum counterexample. Juvan et al. proved [5] that graphs with maximum degree less than 4 are 5-totally choosable. So H is minimally non-7-totally choosable. Lemma 3 implies that $\delta(H) = 4$. Thus H is a 4-regular planar graphs with $2 \times |V|$ edges. By Euler's formula, a planar graph with girth g has at most $\frac{g}{g-2}(|V| - 2)$ edges. Therefore, H contains a triangle, say $C = uvw$. Suppose L is a total-list-assignment of H such that $|L(x)| \geqslant 7$ for any $x \in V \cup E$. Consider the graph $H' = H \setminus \{uv, uw, vw\}$. Then H' has an L-total coloring by the minimality of H. Uncolor u, v and w, then each element incident with C touches at most four colors. This implies that each element incident with C has at least three colors available. It follows from $\chi_l''(C) = \chi''(C) = 3$ that C can be totally colored properly. This implies that H is L-totally colorable, which is a contradiction.

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